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Analytical Bethe ansatz for the boundary $SU(2)$-invariant Thirring model

Yan-Shen Wang

Department of Applied Physics, Xi'an Jiaotong University, Xi'an, 710049, People's Republic of China

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Abstract. The eigenvalue of the transfer matrices for the boundary invariant Thirring model and Bethe ansatz equation satisfied by quasimomenta are obtained by the analytical Bethe ansatz approach.

1. Introduction

One of the most important nonlinear integrable models in one spatial and one time dimension is the $SU(2)$-invariant Thirring model (ITM), with the Lagrangian [1, 2]

$$L = \int \mathrm{d}x\, [\mathrm{i}\bar{\psi}\gamma_\mu \partial_\mu \psi - g(\bar{\psi}\sigma^a\gamma_\mu\psi)(\bar{\psi}\sigma^a\gamma_\mu\psi)] \tag{1}$$

where $\psi = \{\psi_i^\alpha\}$ is the doublet Fermi field (where $i = 1, 2$ is the spinor index and $\alpha = 1, 2$ the isotopic index), Pauli matrices σ^a are the isotropic matrices acting on the isotopic indices, γ_μ are the Dirac matrices: $\gamma_0 = \sigma^x, \gamma_1 = \mathrm{i}\sigma^y, \gamma_5 = \sigma^z$. The model possesses the property of asymptotic freedom. Mass is generated dynamically via the mechanism of dimensional transmutation. Physical excitations are two-component massive particles (kinks). The kinks are parametrized by their rapidities u_i and isotopic $\epsilon = 1, 2$. The S-matrix is given by

$$S_{12}(u) = \begin{bmatrix} u+2c, & & & \\ & u, & 2c, & \\ & 2c, & u, & \\ & & & u+2c \end{bmatrix} = u + 2cP_{12} \tag{2}$$

here $u = u_1 - u_2$ is the difference of rapidities, $c = -\frac{1}{2}\mathrm{i}\pi$ and P_{12} is the permutation matrix. The S-matrix (2) is factorizable, that is to say, it is a solution of the Yang–Baxter equation (YBE) [3],

$$S_{12}(u_1-u_2)S_{13}(u_1-u_3)S_{23}(u_2-u_3) = S_{23}(u_2-u_3)S_{13}(u_1-u_3)S_{12}(u_1-u_2). \tag{3}$$

For the massive theory in the bulk, this equation result in an infinite set of mutually commutative integrals of motion. So the model may be solved exactly by the Bethe ansatz.

Since the work of Cherednik [4], much more interest was focused on the field theory with integrable boundary. It is shown that, in the boundary case, besides the factorizable S-matrix,

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the reflection matrix K^+ and its dual K^-, which satisfy the reflection equation (RE) and the dual RE

$$
\begin{aligned}
&S_{12}(u_1-u_2)K_1^+(u_1)S_{21}(u_1+u_2)K_2^+(u_2) = K_2^+(u_2)S_{12}(u_1+u_2)K_1^+(u_1)S_{21}(u_1-u_2) \\
&S_{12}(-u_1+u_2)K_1^-(u_1)S_{21}(-u_1-u_2-4c)K_2^-(u_2) \\
&\qquad = K_2^-(u_2)S_{12}(-u_1-u_2-4c)K_1^-(u_1)S_{21}(-u_1+u_2)
\end{aligned}
\tag{4}
$$

are necessary to ensure the integrability of the theory [5]. For the ITM with an integrable boundary, a reflection matrix was given in [6],

$$K^+(u) = \begin{bmatrix} u+c+\mu & \\ & -u+c+\mu \end{bmatrix} \tag{5}$$

$$K^-(u) = \begin{bmatrix} -u-c+\nu & \\ & u+3c+\nu \end{bmatrix} \tag{6}$$

where μ and ν are the right and left boundary parameters, respectively.

In integrable systems, an important role is played by a transfer matrix, which may be considered as a generating function for a family of commuting Hamiltonians. The transfer matrix can be constructed by monodromy matrix $T(u)$, which reads

$$
\begin{aligned}
&T(u) = \prod_{j=1}^{N} L(j,u) \\
&L(j,u) = \begin{bmatrix} (u+c)+c\sigma_j^z, & 2c\sigma_j^- \\ 2c\sigma_j^+, & (u+c)-c\sigma_j^z \end{bmatrix}
\end{aligned}
\tag{7}
$$

in the case of the bulk. Here σ_j^3 and $\sigma_j^{\pm} = \frac{1}{2}(\sigma_1 \pm \mathrm{i}\sigma_2)_j$ are isotopic operators acting on the jth isotopic space. $T(u)$ act on $V_a \otimes V_i^{\otimes N}$ $(V_i = C^2)$ with V_a denoting the auxiliary space and $V_i^{\otimes N}$ the 2^N-dimensional isotopic space. $L(j,u)$ can be intertwined by the S-matrix,

$$S_{12}(u-v)L_1(j,u)L_2(j,v) = L_2(j,v)L_1(j,u)S_{12}(u-v) \tag{8}$$

and having transpose symmetry,

$$L^{t_j}(j,u) = QL(j,-u-2c)Q \tag{9}$$

where Q is a 2×2-matrix $Q = -\mathrm{i}\sigma_2 = \left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right]$. The special structure of $S(K^{\pm})$-matrices and L operators completely determine the spectra of the transfer matrices. In this paper, the eigenvalues of the transfer matrix for boundary ITM will be calculated by the analytical Bethe ansatz proposed by Reshetikhin [7].

2. Transfer matrix

From (2), one knows that $S_{12}(u) = S_{21}(u)$, $S_{12}(0) = 2cP_{12}$ and $S_{12}(\pm 2c) = \pm 4cP_{12}^{\pm}$. Here, $P_{12}^{\pm} = \frac{1}{2}(1 \pm P_{12})$ are projective operators. Besides YBE, the S-matrix (2) also satisfies the unitarity relation and crossing symmetry,

$$
\begin{aligned}
&S_{12}(u)S_{21}(-u) = (4c^2-u^2)I_{12} \\
&S_{12}^{t_2}(u) = S_{12}^{t_1}(u) = Q_1 S_{12}(-u-2c)Q_1
\end{aligned}
\tag{10}
$$

where t_1 and t_2 denote the transposition with respect to the first and second spaces, respectively. $Q_1 = Q \otimes 1$. The reflection matrices $K^{\pm}(u)$ also satisfy the unitarity relation,

$$\begin{aligned} &K^+(u)K^+(-u) = (c+\mu)^2 - u^2 \\ &K^-(u-2c)K^-(-u-2c) = (c+\nu)^2 - u^2 \end{aligned} \tag{11}$$

and have the following crossing symmetry:

$$\begin{aligned} &-2(u+2c)Q_i^j K^+(u)_j^k(u)Q_k^l = S_{ki}^{lj}(2u)K^+(-u-2c)_j^k \\ &2uQ_i^j K^-(u)_j^k(u)Q_k^l = S_{ki}^{lj}(-2u-4c)K^-(-u-2c)_j^k. \end{aligned} \tag{12}$$

A transfer matrix in the case of a boundary is defined by [5]

$$\begin{aligned} \tau_a(u) &= \mathrm{tr}_a\, K^-(u)T(u)K^+(u)T^{-1}(-u) \\ &= \mathrm{tr}_a\, K^-(u)L(1,u)\cdots L(N,u)K^+(u)L^{-1}(N,-u)\cdots L^{-1}(1,-u) \end{aligned} \tag{13}$$

where tr_a stands for the trace in the auxiliary space and $L^{-1}(j,u)$ are defined by

$$L^{-1}(j,u) = L(j,-u). \tag{14}$$

Utilizing the property of L and K (equations (8)–(10) and (12)), one can transpose $\tau_a(u)$ with respect to isotopic space,

$$\begin{aligned} \tau_a^t(u) &= \mathrm{tr}_a\, K_a^+(u)L^{-1,t_N}(N,-u)\cdots L^{1,t_1}(1,-u)K_a^-(u)L^{t_1}(1,u)\cdots L^{t_N}(N,u) \\ &= \mathrm{tr}_a\, Q_a K_a^+(u)Q_a L(N,u+2c)\cdots L(1,u+2c) \\ &\quad \times Q_a K_a^-(u)Q_a L(1,-u-2c)\cdots L(N,-u-2c) \\ &= \mathrm{tr}_a\, K_a^-(-u-2c)L(1,-u-2c)\cdots L(N,-u-2c) \\ &\quad \times K_a^+(-u-2c)L^{-1}(N,u+2c)\cdots L^{-1}(1,u+2c) \\ &= \tau_a(-u-2c). \end{aligned} \tag{15}$$

On the other hand, owing to (8), one has

$$P_{12}^- L_1(j,u)L_2(j,u+2c)P^+ = 0. \tag{16}$$

This relation allows one to write the produce $L_1(u)L_2(u+2c)$ as the following block-triangular structure in the auxiliary space $V_1 \otimes V_2$:

$$L_1(u)L_2(u+2c) = \begin{bmatrix} u(u+4c)I_3, & 0 \\ *, & (u+2c)f_{\langle 12\rangle 3}(u) \end{bmatrix} \tag{17}$$

where I_3 is an identity operator in the one-dimensional antisymmetric auxiliary space and $f_{\langle 12\rangle 3}(u)$ is the symmetric fusion of the two S-matrices in the three-dimensional symmetric auxiliary space.

In a similar way, from (4), it follows that

$$\begin{aligned} &P_{12}^- K_1^+(u)S_{21}(2u+2c)K_2^+(u+2c)P_{12}^+ = 0 \\ &P_{12}^- K_2^-(u+2c)S_{12}(-2u-6c)K_1^-(u)P_{12}^+ = 0 \end{aligned} \tag{18}$$

and one can write

$$\begin{aligned} &K_1^+(u)S_{21}(2u+2c)K_2^+(u+2c) \\ &\quad = \begin{bmatrix} -2u(u+3c+\mu)(u+c-\mu), & 0 \\ *, & (u+2c)K_{\langle 12\rangle}^+(u) \end{bmatrix} \end{aligned} \tag{19}$$

$$K_2^-(u+2c)S_{12}(-2u-6c)K_1^-(u)$$
$$= \begin{bmatrix} 2(u+4c)(u+c-\nu)(u+3c+\nu), & 0 \\ *, & (u+2c)K_{\langle 12\rangle}^-(u) \end{bmatrix}. \tag{20}$$

$K_{\langle 12\rangle}^{\pm}(u)$ are the symmetric fusion of $K^+(u)$ or $K^-(u)$, respectively. The block-triangular forms in auxiliary space make it simple to calculate the product of two transfer matrices with spectral parameters u and $u+2c$, respectively,

$$\tau(u)\tau(u+2c) = \frac{u^{2N+1}(u+4c)^{2N+1}}{(u+c)(u+3c)}(u+c-\mu)(u+3c+\mu)(u+c-\nu)(u+3c+\nu)$$
$$-\frac{(u+2c)^{2N+2}}{4(u+c)(u+3c)}F(u) \tag{21}$$

where $F(u)$ is the transfer matrix for the symmetric fused model.

3. Eigenvalue of the transfer matrix

Let $\Lambda(u)$ be the eigenvalue of the transfer matrices, from (15) and (21) one knows that they must satisfy

$$\Lambda(u) = \Lambda(-u-2c) \tag{22}$$
$$\Lambda(u)\Lambda(u+2c) = \frac{u^{2N+1}(u+4c)^{2N+1}}{(u+c)(u+3c)}(u+c-\mu)(u+3c+\mu)(u+c-\nu)(u+3c+\nu)$$
$$-\frac{(u+2c)^{2N+2}}{4(u+c)(u+3c)}f(u). \tag{23}$$

Now we analyse the asymptotic behaviour of $\Lambda(u)$ in the limit $u \to \infty$. Denote $\hat{L}_j(u) = L(j,u) \otimes L^{-1,t_a}(j,-u)$, then

$$\begin{aligned} \tau_a(u) &= \mathrm{tr}_a\, K^-(u)^i_l \hat{T}^{jk}_{il} K^+(u)^k_j \\ \hat{T} &= \hat{L}_1 \cdots \hat{L}_N. \end{aligned} \tag{24}$$

Let $y = u + c$, one has

$$\hat{L}_j(u) \xrightarrow{y\to\infty} y^2 + 2cy(\sigma_j^z\tau_1 + \sigma_j^+\tau_2 + \sigma_j^-\tau_3) + \mathrm{O}(y^0) \tag{25}$$

where τ_i are three 4×4-matrices,

$$\tau_1 = \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{bmatrix} \qquad \tau_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad \tau_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{26}$$

With the help of (25), one can write

$$\hat{T} = \prod_{j=1}^{N} \hat{L}(j,u) \xrightarrow{y\to\infty} y^{2N} + 2cy^{2N-1}(N-M)\tau_1 + 2cy^{2N-1}$$
$$\times\left(\sum_{j=1}^{N}\sigma_j^+\tau_2 + \sum_{j=1}^{N}\sigma_j^-\tau_3\right) + \mathrm{O}(y^{2N-2}) \tag{27}$$

where $M = N - \sum_{i=1}^{N} \sigma_3^{(i)}$. From (24), it follows that M commute with the transfer matrix. Let $|m\rangle$ be the simultaneous eigenstate of $\tau(u)$ and M, $\tau(u)|m\rangle = \Lambda_m(u)|m\rangle$, $M|m\rangle = m|m\rangle$. From (24) and (27), the eigenvalue $\Lambda_m(u)$ in the limit of $y \to \infty$ is obtained,

$$\begin{aligned}\Lambda_m = K^-(u)^i_l \hat{T}^{jk}_{il} K^+(u)^k_j &= a_0 A_0[y^{2N} + 2c(N-m)y^{2N-1} + \mathrm{O}(y^{2N-2})] \\ &\quad \times b_0 B_0[y^{2N} - 2c(N-m)y^{2N-1} + \mathrm{O}(y^{2N-2})].\end{aligned} \tag{28}$$

where

$$\begin{aligned} A_0 &= u + c + \mu \qquad & B_0 &= -u + c + \mu \\ a_0 &= -u - c + \nu \qquad & b_0 &= u + 3c + \nu. \end{aligned} \tag{29}$$

In particular, when $m = 0$, which correspond to all N isotopic spins in the up state, $|0\rangle = \uparrow \otimes \uparrow \otimes \cdots \otimes \uparrow$, one has

$$\begin{aligned}\Lambda_0 &= \langle 0|\tau(u)|0\rangle \\ &= \mathrm{tr}_a\, K_a^-(u)\langle\uparrow|L(1,u)\cdots\langle\uparrow|L(N,u)K_a^+(u)L^{-1}(N,-u)|\uparrow\rangle\cdots L^{-1}(1,-u)|\uparrow\rangle.\end{aligned} \tag{30}$$

Note that $K_a^+(u)$ is a diagonal form, let us show that $\langle\uparrow|L(1,u)\cdots\langle\uparrow|L(N,u)K_a^+(u)$ $L^{-1}(N,-u)|\uparrow\rangle\cdots L^{-1}(1,-u)|\uparrow\rangle$ is also a diagonal form. Suppose $\langle\uparrow|L(j,u)\cdots\langle\uparrow|L(N,u)$ $K_a^+(u)L^{-1}(N,-u)|\uparrow\rangle\cdots L^{-1}(j,-u)|\uparrow\rangle = \begin{pmatrix} A_{N-j+1} & \\ & B_{N-j+1}\end{pmatrix}$, then

$$\begin{aligned}&\langle\uparrow|L(j-1,u)\cdots\langle\uparrow|L(N,u)K_a^+(u)L^{-1}(N,-u)|\uparrow\rangle\cdots L^{-1}(j-1,-u)|\uparrow\rangle \\ &\qquad = \begin{bmatrix} (u+2c)\langle\uparrow| & 0 \\ 2c\langle\downarrow| & u\langle\uparrow| \end{bmatrix}\begin{bmatrix} A_{N-j+1} & 0 \\ 0 & B_{N-j+1}\end{bmatrix}\begin{bmatrix} (u+2c)|\uparrow\rangle & 2c|\downarrow\rangle \\ 0 & u|\uparrow\rangle\end{bmatrix} \\ &\qquad = \begin{bmatrix} A_{N-j+1}(u+2c)^2 & 0 \\ 0 & B_{N-j+1}u^2 + A_{N-j+1}(2c)^2 \end{bmatrix} \\ &\qquad = \begin{bmatrix} A_{N-j+2} & 0 \\ 0 & B_{N-j+2}\end{bmatrix}.\end{aligned} \tag{31}$$

So we have the following successive relations:

$$\begin{aligned} A_{N-j+2} &= (u+2c)^2 A_{N-j+1} \\ B_{N-j+2} &= u^2 B_{N-j+1} + (2c)^2 A_{N-j+1}.\end{aligned} \tag{32}$$

From these relations, we obtain

$$\begin{aligned} A_N &= (u+2c)^{2N} A_0 \\ B_N &= u^{2N} B_0 + (2c)^2 \frac{u^{2N} - (u+2c)^{2N}}{u^2 - (u+2c)^2} A_0.\end{aligned} \tag{33}$$

So Λ_0 reads as follows:

$$\Lambda_0 = (u+2c)^{2N} A_0\left(a_0 + \frac{c}{u+c} b_0\right) + u^{2N} b_0 \left(B_0 - \frac{c}{u+c} A_0\right). \tag{34}$$

The general form of $\Lambda_m(u)$ can be assumed as

$$\begin{aligned}\Lambda_m &= \langle m|\tau(u)|m\rangle \\ &= (u+2c)^{2N} A_0\left(a_0 + \frac{c}{u+c} b_0\right) Q(u) + u^{2N} b_0\left(B_0 - \frac{c}{u+c} A_0\right) P(u).\end{aligned} \tag{35}$$

From (22), one knows that $Q(u)$ and $P(u)$ must satisfy

$$\bar{Q}(u) = P(-u-2c) \qquad \bar{P}(u) = Q(-u-2c) \tag{36}$$

and from equation (23), it follows that

$$Q(u+2c)P(u) = 1 \qquad Q(u)\bar{Q}(-u) = 1 \qquad P(u-2c)\bar{P}(-u-2c) = 1. \tag{37}$$

Owing to (37), $Q(u)$ and $P(u)$ are required to have the following asymptotic behaviour:

$$\begin{aligned} Q(u) &\xrightarrow{y\to\infty} 1 + \frac{1}{y}q + \mathrm{O}(y^{-2}) \\ P(u) &\xrightarrow{y\to\infty} 1 + \frac{1}{y}p(y) + \mathrm{O}(y^{-2}). \end{aligned} \tag{38}$$

Compare (28) with (38), one has

$$q = -2cm \qquad p = 2cm. \tag{39}$$

So $Q(u)$ and $P(u)$ may be written as

$$\begin{aligned} Q(y) &= \prod_{i=1}^{m} \frac{y-2c-\alpha_i}{y-\alpha_i} \qquad \text{or} \qquad Q(u) = \prod_{i=1}^{m} \frac{u-c-\alpha_i}{u+c-\alpha_i} \\ P(u) &= Q^{-1}(u+2c) = \prod_{i=1}^{m} \frac{u+3c-\alpha_i}{u+c-\alpha_i}. \end{aligned} \tag{40}$$

The eigenvalues of the transfer matrices on the m-magnon eigenvector $|m\rangle$ are

$$\begin{aligned} \Lambda_m = {} & (u+2c)^{2N} A_0 \left(a_0 + \frac{c}{u+c} b_0\right) \prod_{i=1}^{m} \frac{u-c-\alpha_i}{u+c-\alpha_i} \\ & + u^{2N} b_0 \left(B_0 - \frac{c}{u+c} A_0\right) \prod_{i=1}^{m} \frac{u+3c-\alpha_i}{u+c-\alpha_i}. \end{aligned} \tag{41}$$

Equation (41) is required to be analyticity, that is to say, the residue of (41) at all poles should vanish. This result in the Bethe ansatz equations

$$\left(\frac{\alpha_i + c}{\alpha_i - c}\right)^{2N+1} = \frac{(\alpha_i + 2c + \nu)(\alpha_i - \mu)}{(\alpha_i - 2c - \nu)(\alpha_i + \mu)} \prod_{j=1,\neq i}^{m} \frac{\alpha_i - \alpha_j + 2c}{\alpha_i - \alpha_j - 2c}. \tag{42}$$

4. Discussion

In this paper, the approach of the analytical Bethe ansatz is applied to the boundary $SU(2)$-invariant Thirring model, the eigenvalue of the transfer matrices and the Bethe ansatz equation satisfied by the quasimomenta are obtained. This approach does not depend on whether a special pseudovacuum exists, on which the L operators act as triangular matrices. It only utilizes the structure of the scattering and reflection matrix. So it may be applied to more general cases, such as a boundary sine–Gordon model, for which the construction of multiparticle states is more complicated. The results for the boundary sine–Gordon model are expected to be similar to those of the boundary ITM, because ITM may be considered as a limit of the sine–Gordon model.

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